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MATHEMATICAL MODELS FOR USE IN THE READJUSTMENT OF THE NORTH AMERICAN GEODETIC NETWORKS

by

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TABLE OF CONTENTS

0. ABSTRACT

A comprehensive description of the mathematical models which will be used in the readjustment of the North American "horizontal" networks is given. These models have their basis in three dimensional geodetic coordinate systems; stations for which there is no (or very weak) information in the vertical direction have their ellipsoidal heights held fixed. This approach has been extensively developed by Vincenty and Bowring (1978) and Vincenty (1979, 1980a, 1980b, 1982a, 1982b) and is extended here to include treatment of inertial surveys. Also given here is a discussion of auxiliary parameters to be included in the readjustment for modelling systematic error effects.

RÉSUMÉ

Une description simple des modèles mathématiques qui seront utilisés pour la redéfinition des réseaux "horizontaux" de l'Amérique du Nord est présentée. Ces modèles ont leur fondement dans les systèmes géodésiques tridimensionnels; les stations pour lesquelles il n'y a pas (ou peu) d'information dans la direction verticale, ont leur élévation ellipsoïdique tenue fixe. Cette approche a été largement développée par Vincenty et Bowring (1978) et Vincenty (1979, 1980a, 1980b, 1982a, 1982b) et est ici étendue pour inclure les données du positionnement inertiel. Une discussion des paramètres auxilières à inclure dans la redéfinition pour modeler les effets des erreurs systématiques est donnée.

1. INTRODUCTION

The classical method of adjusting horizontal geodetic networks involves reducing the terrestrial distance, unoriented direction and astronomic azimuth observations to a "best fitting" reference ellipsoid for the region of interest and then performing computations on this reference surface. When information in the vertical direction is non-existent or very weak, this approach serves our purposes very well. Another approach, however, is to formulate relationships among the observations, as referred to the station markers themselves, and coordinates in a three dimensional coordinate system. In this case, when position information in the vertical direction is very weak for a particular station, the coordinate in the vertical direction can be held fixed in the adjustment. Even when vertical information is very weak for all stations involved, this approach turns out to be simpler than the classical approach, both conceptually and computationally. When vertical information is relatively strong over lines of arbitrary length, as with satellite Doppler and Very Long Baseline Interferometric (VLBI) observations, all three coordinates can easily be included as unknowns in the adjustment. With this approach we not only have a simple and correct treatment of the three-dimensional information but also have the adjusted network automatically referred to the chosen reference system in an optimum way.

1.1 Summary of Notation

For convenience, we give a summary of notation to be used herein:

ϕ	-	geodetic latitude (positive north of the equator)
λ	-	geodetic longitude (positive east of the Greenwich meridian)
h	-	height above the reference ellipsoid measured along the ellipsoidal normal
Φ	-	astronomic latitude (positive north of the equator)
Λ	-	astronomic longitude (positive east of the Greenwich meridian)
H	-	orthometric height
N	-	geoidal height above the reference ellipsoid
ξ	-	meridian component of the surface deflection of the vertical
η	-	prime vertical component of the surface deflection of the vertical
ρ	-	meridian radius of curvature of the reference ellipsoid (p.26)
ν	-	prime vertical radius of curvature of the reference ellipsoid (p.4)
a	-	major semi axis of the reference ellipsoid
b	-	minor semi axis of the reference ellipsoid
e	-	first eccentricity of the reference ellipsoid ($e^2 = 1 - b^2/a^2$)
A	-	astronomic azimuth (clockwise from north)
d	-	unoriented horizontal direction
Ω	-	orientation unknown for a set of unoriented horizontal directions
S	-	spatial straight-line interstation distance
X,Y,Z	-	geocentric C.T. coordinates
x,y,z	-	topocentric local geodetic coordinates
u,v,w	-	topocentric local astronomic coordinates
C.T.	-	used to denote the Conventional Terrestrial coordinate system

L.G. - used to denote local geodetic coordinate systems

L.A. - used to denote local astronomic coordinate systems

≐ - approximately equal to

1.2 Observables and Parameters

Before we consider the development of the observation equations to be used in the NAD83 adjustment, a brief discussion of which observations are to be included, as observations, will be helpful. That is, we can treat observational information in one of two ways:

i) by writing an observation equation for the observation which is to be included in the adjustment along with its finite, non-zero weight (i.e. treated as an observation) or

ii) by using the observation as supplementary information, i.e. as a constant to provide information needed for coordinate transformations, etc. (e.g. as the observed astronomic latitudes and longitudes will be used in the readjustment).

It is realized that the major objective of our efforts in the readjustment of the continental networks is to determine, insofar as possible, a set of distortion-free coordinates for our geodetic stations. We are ultimately interested in having these coordinate estimates for all three dimensions and one may naturally ask why we do not combine all our measurements in a three dimensional adjustment, thus producing three dimensional coordinates for each network station. That this is theoretically possible has, of course, been known for a long time. But especially now that we have observations such as satellite Doppler, which are giving three dimensional positional information, is a three dimensional adjustment not now desirable from a practical point of view? Do we have something to gain from such an approach?

There is no question that terrestrial observations of distance, unoriented direction and azimuth will be included in the readjustment. Satellite Doppler determined coordinates and coordinate differences will also be included. The above questions really refer to vertical angles and orthometric heights. The reason why orthometric heights will not be included in the readjustment as observations is that these have only been observed for a very small portion of the horizontal network. The reason why vertical angles will not be included is that they are in general of poor quality and contain unknown systematic errors due to refraction. These are, of course, general statements and there may indeed be areas where the inclusion, as observations, of observed orthometric heights or observed vertical angles will be of some benefit. The fact is, however, that these observations will not be included (as observations) in the continental readjustment. Further study may support their inclusion in future adjustments.

Similarly, since there is as yet no demonstrated benefit to be gained by including geoidal undulations and deflections of the vertical as observations in the readjustment, these will also only be included as supplementary information. What is meant by this will become clear in our development of the mathematical models in the following sections. Since the vertical angles and deflections will not be included as observations, we will not include the observed astronomic latitudes and longitudes as observations.

We are left with the following types of observables for which observation equations will be included in the readjustment:

- unoriented horizontal directions
- station-to-station spatial straight-line distances
- astronomic azimuths
- satellite Doppler three dimensional coordinates and coordinate differences
- Very Long Baseline Interferometric coordinate differences
- differences in ellipsoidal latitude and longitude determined by Inertial Survey Systems (ISS)

It is possible that the last two types of observations (VLBI and ISS) may not be prepared in time for inclusion. In any case, we see that, for the purposes of the readjustment, only stations having Doppler or VLBI observations associated with them have observational information supplied in all three dimensions. For each of these stations, there will be three coordinate unknowns. For the other stations, however, which only have information supplied in two dimensions, by the (horizontal) terrestrial or ISS observations, there can be only two (horizontal) coordinate unknowns. Other unknown parameters to be determined in the adjustment such as scale and orientation unknowns will be discussed in detail in following sections.

1.3 Coordinate Systems and their Relationship

In this section we give a brief review of the coordinate systems which form the mathematical basis of the observation equations to be developed in section 2. These coordinate systems are

i) the Conventional Terrestrial system,
ii) the local astronomic system,
iii) the local geodetic system, and
iv) the ellipsoidal system.

The Conventional Terrestrial (C.T.) system is a three dimensional Cartesian system with its origin at the geocentre. The Z-axis is defined by the Conventional International Origin (CIO), (i.e. the average position of Earth's rotation axis during the years 1900-1905). The C.T. X-Z plane is parallel to the Mean Greenwich Zero Meridian and the system is right handed. The ellipsoidal system with which we are concerned (which has been adopted by the I.A.G. as part of Reference System 1980) consists of an ellipsoid of revolution whose geometric centre is coincident with the geocentre and whose minor semi-axis is coincident with the C.T. Z-axis. The size of this reference ellipsoid is given by its major semi-axis (equatorial radius) a = 6378137.000 metres and minor semi-axis b = 6356752.314 metres. Figure 1.1 illustrates the relationship of the ellipsoidal system with the C.T. system. The mathematical relationship of C.T. X, Y, Z coordinates with ϕ, λ, h is given by

$$X = (\nu + h) \cos \phi \cos \lambda \tag{1.1}$$

$$Y = (\nu + h) \cos \phi \sin \lambda \tag{1.2}$$

$$Z = (\nu(1-e^2) + h) \sin \phi \tag{1.3}$$

where

$$\nu = a/(1-e^2 \sin^2\phi)^{1/2} \quad . \tag{1.4}$$

See p. 20. for ρ_i

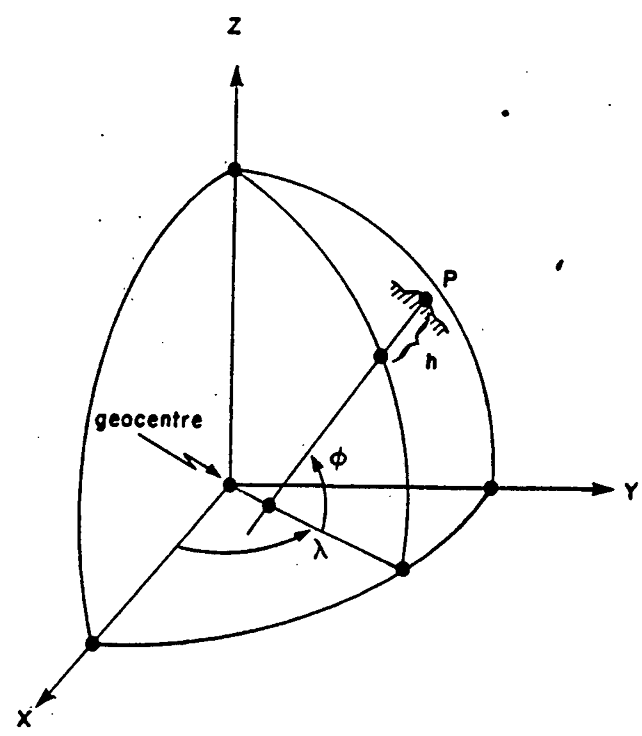

FIGURE 1.1: ELLIPSOIDAL AND C.T. SYSTEMS

The other two coordinate systems we need are topocentric systems. By "topocentric" is meant a system with its origin at the station marker itself. The w-axis of the local astronomic (L.A.) system is coincident with the gravity vector at the station and is positive upwards. The L.A. u-axis points to astronomic north and the L.A. v-axis points to astronomic east. The relationship of the L.A. system with the C.T. system is illustrated in Figure 1.2. Note that the astronomic latitude Φ and astronomic longitude Λ are assumed to be corrected for polar motion (i.e. reduced to the CIO) throughout this paper. For a L.A. system at station i, whose astronomic latitude and longitude are Φ_i and Λ_i, we will denote the L.A. coordinates of a second point j by

(u_{ij}, v_{ij}, w_{ij}). The relationship of these L.A. coordinates with C.T. coordinate differences is given by

$$\begin{vmatrix} u_{ij} \\ v_{ij} \\ w_{ij} \end{vmatrix} = \begin{vmatrix} -\sin\Phi_i\cos\Lambda_i & -\sin\Phi_i\sin\Lambda_i & \cos\Phi_i \\ -\sin\Lambda_i & \cos\Lambda_i & 0 \\ \cos\Phi_i\cos\Lambda_i & \cos\Phi_i\sin\Lambda_i & \sin\Phi_i \end{vmatrix} \begin{vmatrix} \Delta X_{ij} \\ \Delta Y_{ij} \\ \Delta Z_{ij} \end{vmatrix} \quad (1.5)$$

where $\Delta X_{ij} = X_j - X_i$, $\Delta Y_{ij} = Y_j - Y_i$ and $\Delta Z_{ij} = Z_j - Z_i$.

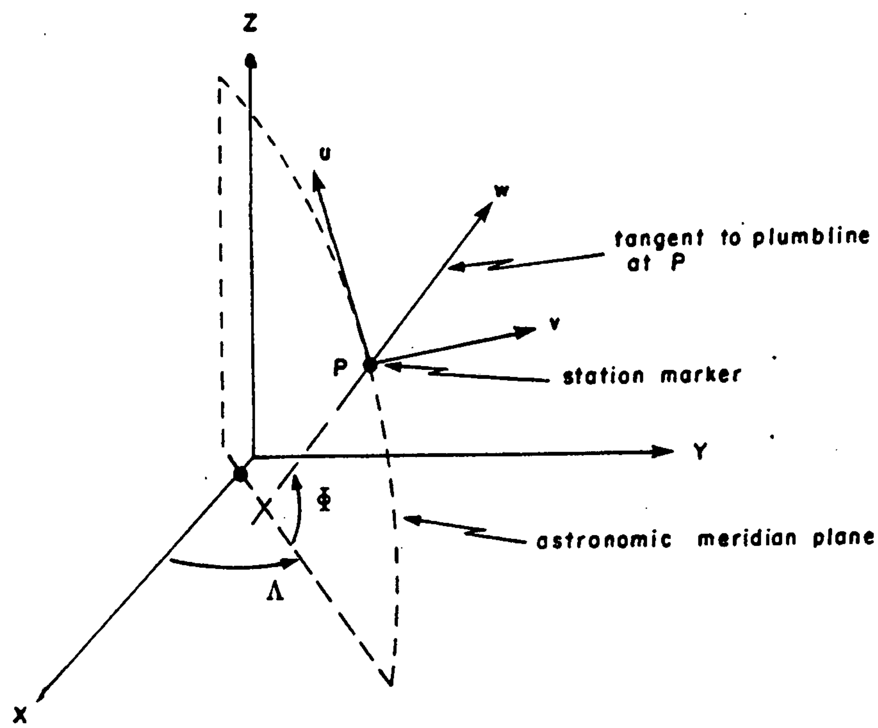

FIGURE 1.2: LOCAL ASTRONOMIC SYSTEM

The z-axis of the local geodetic (L.G.) system is coincident with the ellipsoidal normal at the station and is positive upwards. The L.G. x-axis points to geodetic north and the L.G. y-axis points to geodetic east. The relationship of the L.G. system with the C.T. system is illustrated in Figure 1.3.

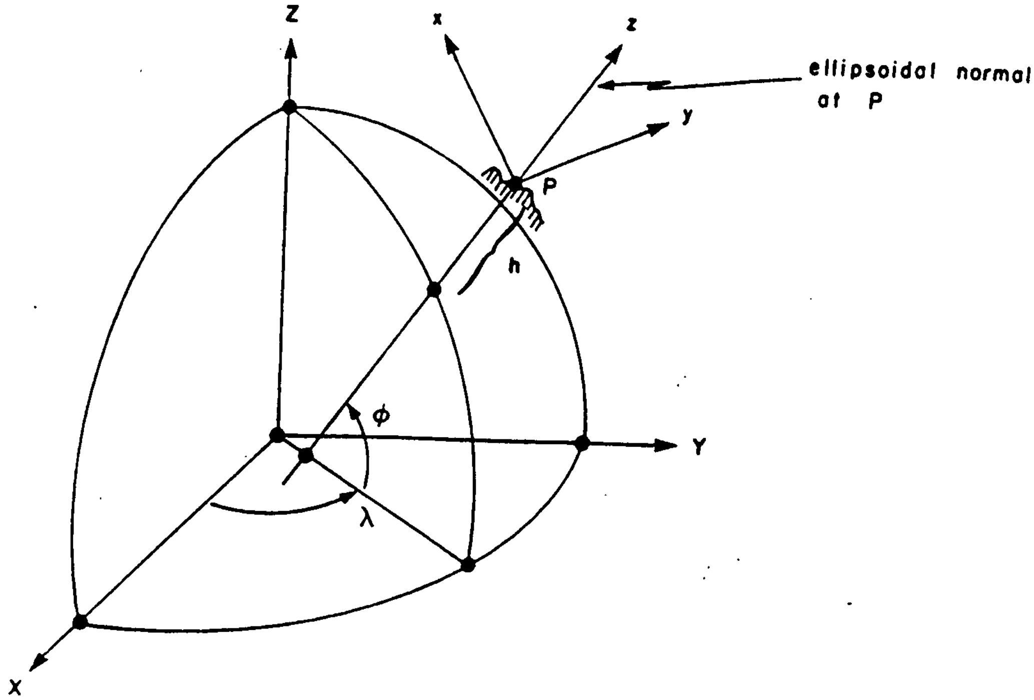

FIGURE 1.3: LOCAL GEODETIC SYSTEM

The relationship of the L.G. coordinates (x_{ij}, y_{ij}, z_{ij}) (where we use the same subscript notation as for the L.A. coordinates) with C.T. coordinate differences is given by

$$\begin{vmatrix} x_{ij} \\ y_{ij} \\ z_{ij} \end{vmatrix} = \begin{vmatrix} -\sin\phi_i\cos\lambda_i & -\sin\phi_i\sin\lambda_i & \cos\phi_i \\ -\sin\lambda_i & \cos\lambda_i & 0 \\ \cos\phi_i\cos\lambda_i & \cos\phi_i\sin\lambda_i & \sin\phi_i \end{vmatrix} \begin{vmatrix} \Delta X_{ij} \\ \Delta Y_{ij} \\ \Delta Z_{ij} \end{vmatrix} . \qquad (1.6)$$

Another relationship we will need is that of the L.A. and L.G. systems. If we let

$$R_G = \begin{vmatrix} -\sin\phi_i\cos\lambda_i & -\sin\phi_i\sin\lambda_i & \cos\phi_i \\ -\sin\lambda_i & \cos\lambda_i & 0 \\ \cos\phi_i\cos\lambda_i & \cos\phi_i\sin\lambda_i & \sin\phi_i \end{vmatrix} \qquad (1.7)$$

and

$$R_A = \begin{vmatrix} -\sin\Phi_i\cos\Lambda_i & -\sin\Phi_i\sin\Lambda_i & \cos\Phi_i \\ -\sin\Lambda_i & \cos\Lambda_i & 0 \\ \cos\Phi_i\cos\Lambda_i & \cos\Phi_i\sin\Lambda_i & \sin\Phi_i \end{vmatrix} \qquad (1.8)$$

then, from equation (1.6),

$$\begin{vmatrix} x \\ y \\ z \end{vmatrix} = R_G \begin{vmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{vmatrix} \tag{1.9}$$

and, from equation (1.5),

$$\begin{vmatrix} u \\ v \\ w \end{vmatrix} = R_A \begin{vmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{vmatrix} \tag{1.10}$$

where we have dropped the i, j subscripts for convenience. Since the matrix R_A is orthogonal, we have $R_A^{-1} = R_A^{+}$ (i.e. its inverse is given by its transpose) and thus we can write

$$\begin{vmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{vmatrix} = R_A^{+} \begin{vmatrix} u \\ v \\ w \end{vmatrix} . \tag{1.11}$$

Substituting this in equation (1.9) gives

$$\begin{vmatrix} x \\ y \\ z \end{vmatrix} = R_G R_A^{+} \begin{vmatrix} u \\ v \\ w \end{vmatrix} . \tag{1.12}$$

which expresses the relationship of the L.G. and L.A. coordinate systems. The product $R_G R_A^{+}$ is given to a sufficient degree of accuracy for our purposes by

$$R_G R_A^{+} \doteq \begin{vmatrix} 1 & \sin\Phi^*(\Lambda-\lambda) & (\Phi-\phi) \\ -\sin\Phi^*(\Lambda-\lambda) & 1 & \cos\Phi^*(\Lambda-\lambda) \\ -(\Phi-\phi) & -\cos\Phi^*(\Lambda-\lambda) & 1 \end{vmatrix} \tag{1.13}$$

where $(\Phi-\phi)$ and $(\Lambda-\lambda)$ are expressed in radians.

The above relationships of the various coordinate systems form the basis for our derivations of observation equations in the next section.

2. DERIVATION OF THE MATHEMATICAL MODELS

We will first consider observation equations for the terrestrial observations of unoriented direction, spatial distance and astronomic azimuth. We then consider observation equations for satellite Doppler and VBLI observations and finally those for ISS observations.

2.1 Observation Equations for Terrestrial Observations

The terrestrial observations of unoriented direction, spatial distance and astronomic azimuth contain only very weak information in the vertical direction. Therefore, we must restrict our solution of coordinates for the corresponding terrestrial stations (for which there is no other type of measurement, e.g. satellite Doppler) to two dimensions. We must, however, provide information in the vertical direction for these stations in order to arrive at a solution in the other two dimensions. This vertical information must have a definite geometrical relationship with the C.T. coordinate system (since we wish to combine observations of (X, Y, Z) with the terrestrial observations in the same overall model) and can therefore be supplied by ellipsoidal heights (see equations (1.1), (1.2) and (1.3)). It is obvious then, that we wish to treat the ellipsoidal heights of these terrestrial stations as fixed or constant.

We must now decide which coordinate system we will choose in which to solve for the two "horizontal" coordinates of the terrestrial stations. We have decided to hold the ellipsoidal heights of these stations fixed and it may seem that we should formulate our models (observation equations) in terms of (ϕ, λ) (which is the classical approach). It turns out to be simpler, however, to express our models in terms of either local astronomic (u, v) or local geodetic (x, y) coordinates. If we solve for corrections $(\delta u, \delta v)$ to local astronomic u, v coordinates, however, we are, in theory, at the same time affecting a corresponding change δh in the ellipsoidal height (see Figure 2.1). We will therefore formulate our models in terms of local geodetic (x, y) coordinates, which will maintain our fixed ellipsoidal heights. The changes $(\delta x, \delta y)$ computed in the adjustment can easily be converted to corresponding changes $(\delta\phi, \delta\lambda)$ so that the adjustment will result in adjusted latitudes and longitudes (and heights for Doppler stations). This conversion is given by

$$\delta\phi_i = \delta x_i/(\rho_i+h_i)$$

$$\delta\lambda_i = \delta y_i/((\nu_i+h_i)\cos\phi_i) \quad .$$

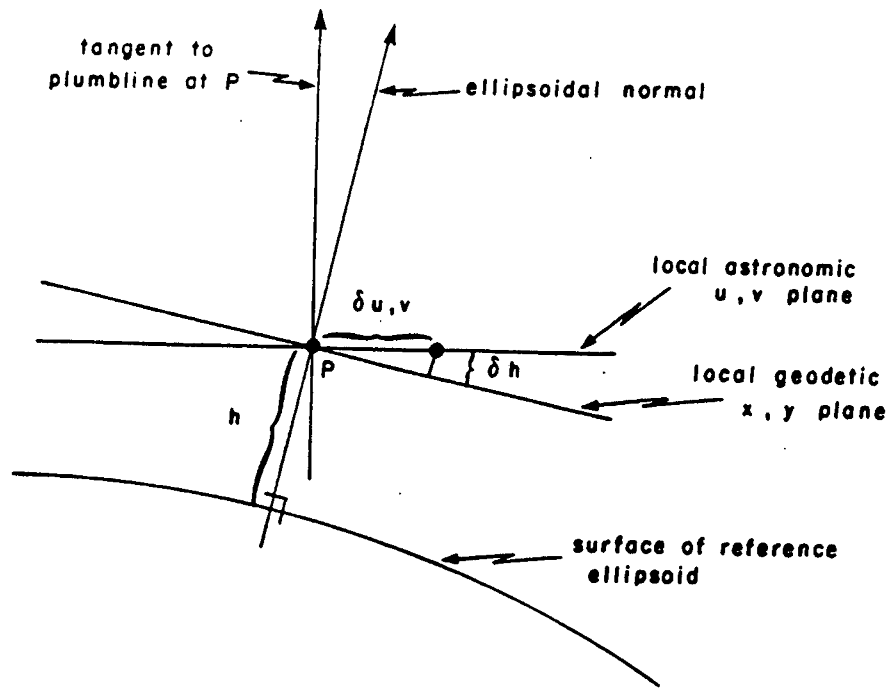

FIGURE 2.1 CHANGE IN ELLIPSOIDAL HEIGHT RESULTING FROM A SHIFT IN THE LOCAL ASTRONOMIC SYSTEM

For the derivation of the observation equations, we will first express them in the C.T. system. In doing this, we are able to include the height information implicitly. Next we will transform the observation equations to the local astronomic systems which allows an implicit inclusion of the observed astronomic latitudes and longitudes. For stations at which astronomic latitude and/or longitude have not been observed, we must compute these quantities using the preliminary ϕ, λ and the deflections of the vertical (see equations (4.7) and (4.8)). Up to this stage in our development, we will have included the geoidal information: geoidal heights via the ellipsoidal heights ($h = H+N$) and deflections of the vertical via computing Φ, Λ. Finally we transform the observation equations from the local astronomic systems to the local geodetic systems. At this stage, for the astronomic azimuth observation equations, we will have included in our equations all the information and coordinate system relationships which form the basis of the classical Laplace azimuth equation. Therefore, although the Laplace equation does not appear explicitly in our models, it is included in an implicit way.

2.1.1 Terrestrial Observation Equations in the C.T. System

We will treat the astronomic azimuth and spatial distance observation equations here. The observation equation for unoriented directions is very similar to that for azimuths and is treated separately in Section 4 of this paper. We note that our treatment of auxiliary and nuisance parameters is dealt with in Sections 3 and 4; these parameters are not considered in this section.

We start with the azimuth equation. Since an astronomic azimuth A_{ij} is measured in the local astronomic (u, v) plane, we can write the functional relationship

$$f_A = \arctan\left\{\frac{v_{ij}}{u_{ij}}\right\} - A_{ij} = 0 \ . \tag{2.1}$$

From equation (1.5) we have

$$u_{ij} = -\sin\Phi_i\cos\Lambda_i(X_j-X_i) - \sin\Phi_i\sin\Lambda_i(Y_j-Y_i) + \cos\Phi_i(Z_j-Z_i) \tag{2.2}$$

and

$$v_{ij} = -\sin\Lambda_i(X_j-X_i) + \cos\Lambda_i(Y_j-Y_i) \ . \tag{2.3}$$

Substituting these in equation (2.1) and differentiating with respect to X_i, Y_i, Z_i, X_j, Y_j and Z_j gives the linearized observation equation (in the C.T. system),

$$r_A = \frac{\partial f_A}{\partial X_i}(\delta X_i - \delta X_j) + \frac{\partial f_A}{\partial Y_i}(\delta Y_i - \delta Y_j) \tag{2.4}$$

$$+ \frac{\partial f_A}{\partial Z_i}(\delta Z_i - \delta Z_j) + t_A$$

where r_A is the residual, t_A is the constant term of the Taylors' series expansion (misclosure) and

$$\frac{\partial f_A}{\partial X_i} = \frac{u_{ij}\sin\Lambda_i - v_{ij}\sin\Phi_i\cos\Lambda_i}{u_{ij}^2 + v_{ij}^2} \tag{2.5}$$

$$\frac{\partial f_A}{\partial Y_i} = \frac{-u_{ij}\cos\Lambda_i - v_{ij}\sin\Phi_i\sin\Lambda_i}{u_{ij}^2 + v_{ij}^2} \tag{2.6}$$

$$\frac{\partial f_A}{\partial Z_i} = \frac{v_{ij}\cos\Phi_i}{u_{ij}^2 + v_{ij}^2} \ . \tag{2.7}$$

We now consider the observation equation for a spatial straight-line distance S_{ij} between two station markers. In this case, we can write the observation equation directly in the C.T. system as

$$f_S = \{(X_j-X_i)^2 + (Y_j-Y_i)^2 + (Z_j-Z_i)^2\}^{1/2} - S_{ij} = 0 \tag{2.8}$$

and its linearized form is given by

$$r_S = \frac{\Delta X_{ij}}{S_{ij}}(\delta X_j - \delta X_i) + \frac{\Delta Y_{ij}}{S_{ij}}(\delta Y_j - \delta Y_i) \qquad (2.9)$$

$$+ \frac{\Delta Z_{ij}}{S_{ij}}(\delta Z_j - \delta Z_i) + t_S$$

where $S_{ij} = (\Delta X^2_{ij} + \Delta Y^2_{ij} + \Delta Z^2_{ij})^{1/2}$.

2.1.2 Terrestrial Observation Equations in the Local Astronomic Systems

We will now transform equations (2.4) and (2.9) from the C.T. system to the local astronomic systems at stations i and j. To do this, we make use of equation (1.11) which is rewritten here in terms of small coordinate changes at station i:

$$\delta X_i = -\sin\Phi_i \cos\Lambda_i \delta u_i - \sin\Lambda_i \delta v_i + \cos\Phi_i \cos\Lambda_i \delta w_i \qquad (2.10)$$

$$\delta Y_i = -\sin\Phi_i \sin\Lambda_i \delta u_i + \cos\Lambda_i \delta v_i + \cos\Phi_i \sin\Lambda_i \delta w_i \qquad (2.11)$$

$$\delta Z_i = \cos\Phi_i \delta u_i + \sin\Phi_i \delta w_i \quad . \qquad (2.12)$$

Substituting these expression in equations (2.4) and (2.9), for both station i and station j, and collecting terms gives the linearized observation equations in the L.A. systems as follows.

For azimuths (which results we can use for directions also) we get

$$r_A = c_1 \delta u_i + c_2 \delta v_i + c_5 \delta w_i + c_3 \delta u_j + c_4 \delta v_j + c_6 \delta w_j + f_A \qquad (2.13)$$

where

$$c_1 = \frac{v_{ij}}{u^2_{ij} + v^2_{ij}} \qquad (2.14)$$

$$c_2 = \frac{-u_{ij}}{u^2_{ij} + v^2_{ij}} \qquad (2.15)$$

$$c_5 = 0 \qquad (2.16)$$

$$c_3 = \frac{-u_{ij}\sin\Phi_j \sin\Delta\Lambda - v_{ij}(\sin\Phi_i \sin\Phi_j \cos\Delta\Lambda + \cos\Phi_i \cos\Phi_j)}{u^2_{ij} + v^2_{ij}} \qquad (2.17)$$

$$c_4 = \frac{u_{ij}\cos\Delta\Lambda - v_{ij}\sin\phi_i\sin\Delta\Lambda}{u_{ij}^2 + v_{ij}^2} \tag{2.18}$$

$$c_6 = \frac{u_{ij}\cos\phi_j\sin\Delta\Lambda + v_{ij}(\sin\phi_i\cos\phi_j\cos\Delta\Lambda - \cos\phi_i\sin\phi_j)}{u_{ij}^2 + v_{ij}^2} \tag{2.19}$$

where

$$\Delta\Lambda = \Lambda_j - \Lambda_i \ . \tag{2.20}$$

For distances, we get

$$r_S = g_1\delta u_i + g_2\delta v_i + g_5\delta w_i + g_3\delta u_j + g_4\delta v_j + g_6\delta w_j + f_S \tag{2.21}$$

where,

$$g_1 = \frac{-u_{ij}}{S_{ij}} \tag{2.22}$$

$$g_2 = \frac{-v_{ij}}{S_{ij}} \tag{2.23}$$

$$g_5 = \frac{-w_{ij}}{S_{ij}} \tag{2.24}$$

$$g_3 = \frac{-u_{ji}}{S_{ij}} \tag{2.25}$$

$$g_4 = \frac{-v_{ji}}{S_{ij}} \tag{2.26}$$

$$g_6 = \frac{-w_{ji}}{S_{ij}} \ . \tag{2.27}$$

Note that although we give the coefficients above for δw_i and δw_j, these coordinate shifts cannot, in general, be determined on the basis of the terrestrial observations. This is, of course, due to the fact that the terrestrial observations provide no or very weak information in the vertical direction (as indicated by the fact that $c_5 = 0$ and $c_6 \doteq g_5 \doteq g_6 \doteq 0$ when vertical angles of lines between stations are small, as is the case). For this reason, we will not include these unknowns (δw_i, δw_j) in terrestrial observation equations.

2.1.3 Terrestrial Observation Equations in the Local Geodetic Systems

In order to transform equations (2.13) and (2.21) from the L.A. systems to the L.G. systems, we will make use of equations (1.12) and (1.13). From those equations, we get (with $\delta z_i = 0$)

$$\delta u_i \doteq \delta x_i - \sin\Phi_i(\Lambda-\lambda)\delta y_i \quad (2.28)$$

$$\delta v_i \doteq \sin\Phi_i(\Lambda-\lambda)\delta x_i + \delta y_i \quad (2.29)$$

For computational purposes we can simplify the coefficient $\sin\Phi_i\cdot(\Lambda-\lambda)$, which is an approximation of

$$m_i = \sin\Phi_i \sin(\Lambda-\lambda) , \quad (2.30)$$

as follows. With the identity

$$\sin(\Lambda-\lambda) = \sin\Lambda\cos\lambda - \cos\Lambda\sin\lambda$$

we have

$$m_i = \sin\Phi_i(\sin\Lambda_i \cdot \frac{X_i}{p_i} - \cos\Lambda\cdot\frac{Y_i}{p_i}) \quad (2.31)$$

where we have sutstituted X_i/p_i for $\cos\lambda_i$ and Y_i/p_i for $\sin\lambda_i$ with

$$p_i = (X_i^2 + Y_i^2)^{1/2} . \quad (2.32)$$

Equations (2.28) and (2.29) can be rewritten as

$$\delta u_i \doteq \delta x_i - m_i\delta y_i \quad (2.33)$$

$$\delta v_i \doteq m_i\delta x_i + \delta y_i . \quad (2.34)$$

Substituting these in equation (2.13), and dropping the δw_i and δw_j unknowns, gives (for azimuths)

$$r_A = \alpha_1\delta x_i + \alpha_2\delta y_i + \alpha_3\delta x_j + \alpha_4\delta y_j + f_A \quad (2.35)$$

where

$$\alpha_1 = c_1 + m_i c_2 \quad (2.36)$$

$$\alpha_2 = c_2 - m_i c_1 \quad (2.37)$$

$$\alpha_3 = c_3 + m_j c_4 \quad (2.38)$$

$$\alpha_4 = c_4 - m_j c_3 \quad . \tag{2.39}$$

Similarly, substituting equations (2.33) and (2.34) in equation (2.21) gives, for distances,

$$r_S = \beta_1 \delta x_i + \beta_2 \delta y_i + \beta_3 \delta x_j + \beta_4 \delta y_j + f_S \tag{2.40}$$

where

$$\beta_1 = g_1 + m_i g_2 \tag{2.41}$$

$$\beta_2 = g_2 - m_i g_1 \tag{2.42}$$

$$\beta_3 = g_3 + m_j g_4 \tag{2.43}$$

$$\beta_4 = g_4 - m_j g_3 \quad . \tag{2.44}$$

2.2 Observation Equations for Satellite Doppler and VLBI Observations

The basic relationship between observed "nearly C.T." Cartesian coordinates X^{obs}, Y^{obs}, Z^{obs} and C.T. coordinates X, Y, Z is given by the seven-parameter similarity transformation

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} X_o \\ Y_o \\ Z_o \end{vmatrix} + (1+k)R_\omega \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix} \tag{2.45}$$

where X_o, Y_o, Z_o are small translations, k is a small scale change and, for small rotations ω_X, ω_Y, ω_Z,

$$R_\omega \doteq \begin{vmatrix} 1 & \omega_Z & -\omega_Y \\ -\omega_Z & 1 & \omega_X \\ \omega_Y & -\omega_X & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & \omega_Z & -\omega_Y \\ -\omega_Z & 0 & \omega_X \\ \omega_Y & -\omega_X & 0 \end{vmatrix}$$

$$= I + U_\omega \ . \qquad (2.46)$$

Equation (2.45) forms the basis for both coordinate and coordinate difference observations. We can simplify the use of this equation as an observation equation by taking advantage of the differential character of these rotation angles, translations and scale change. Considering the differential rotations by themselves, we get

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = R_\omega \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix}$$

$$= (I+U_\omega) \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix} \qquad (2.47)$$

$$= \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix} + U_\omega \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix}$$

or, since

$$\begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix} \doteq \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} \qquad (2.48)$$

$$\begin{vmatrix} X-X^{obs} \\ Y-Y^{obs} \\ Z-Z^{obs} \end{vmatrix} \doteq U_\omega \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = U_r \begin{vmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{vmatrix} \qquad (2.49)$$

where,

$$U_r = \begin{vmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{vmatrix} . \qquad (2.50)$$

Considering differential translations by themselves gives

$$\begin{vmatrix} X-X^{obs} \\ Y-Y^{obs} \\ Z-Z^{obs} \end{vmatrix} = \begin{vmatrix} X_o \\ Y_o \\ Z_o \end{vmatrix} \qquad (2.51)$$

and for differential scale change

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = (1+k) \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix}$$

$$= \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix} + k \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix} \qquad (2.52)$$

or,

$$\begin{vmatrix} X-X^{obs} \\ Y-Y^{obs} \\ Z-Z^{obs} \end{vmatrix} \doteq k \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} \qquad (2.53)$$

Combining all three effects of small rotations, translations and scale change, we get the simplified form

$$\begin{vmatrix} X-X^{obs} \\ Y-Y^{obs} \\ Z-Z^{obs} \end{vmatrix} \doteq \begin{vmatrix} X_o \\ Y_o \\ Z_o \end{vmatrix} + U_r \begin{vmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{vmatrix} + k \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} \tag{2.54}$$

or

$$f_{X,Y,Z} = \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} - \begin{vmatrix} X_o \\ Y_o \\ Z_o \end{vmatrix} - U_r \begin{vmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{vmatrix} - k \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} - \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix} \tag{2.55}$$

which we can use as our observation equation for X^{obs}, Y^{obs}, Z^{obs}. The linearized form of equation (2.55) is

$$\begin{vmatrix} r_X \\ r_Y \\ r_Z \end{vmatrix} = \begin{vmatrix} \delta X \\ \delta Y \\ \delta Z \end{vmatrix} - \begin{vmatrix} \delta X_o \\ \delta Y_o \\ \delta Z_o \end{vmatrix} - U_r \begin{vmatrix} \delta\omega_X \\ \delta\omega_Y \\ \delta\omega_Z \end{vmatrix} - \delta k \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} + f_{X,Y,Z} \tag{2.56}$$

where the misclosure vector $f_{X,Y,Z}$ is given by equation (2.55) evaluated at up-to-date values of the parameters. Vincenty (1982a) has shown that it is sufficient to compute the misclosure vector using

$$f_{X,Y,Z} \doteq \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} - \begin{vmatrix} X^{obs} \\ Y^{obs} \\ Z^{obs} \end{vmatrix} . \tag{2.57}$$

So far, we have expressed the observation equation for X^{obs}, Y^{obs}, Z^{obs} in terms of the coordinate shifts (unknowns) δX, δY, δZ. Since we wish to solve for local geodetic coordinate shifts δx, δy, δz (or δh), we substitute, using equation (1.6),

$$\begin{vmatrix} \delta_x \\ \delta_y \\ \delta h \end{vmatrix} = \begin{vmatrix} -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{vmatrix} \begin{vmatrix} \delta_X \\ \delta_Y \\ \delta_Z \end{vmatrix} \qquad (2.58)$$

for δX, δY, δZ in equation (2.56). Details are given in section 4.

The treatment of "near C.T." coordinate difference observations is very similar. The basic relationship, analogous to equation (2.45), is

$$\begin{vmatrix} \Delta X_{ij} \\ \Delta Y_{ij} \\ \Delta Z_{ij} \end{vmatrix} = (1+k)R_\omega \begin{vmatrix} \Delta X^{obs}_{ij} \\ \Delta Y^{obs}_{ij} \\ \Delta Z^{obs}_{ij} \end{vmatrix} . \qquad (2.59)$$

The resulting observation equation, analogous to equation, (2.55) is given by

$$\begin{vmatrix} r_{\Delta X_{ij}} \\ r_{\Delta Y_{ij}} \\ r_{\Delta Z_{ij}} \end{vmatrix} = - \begin{vmatrix} \delta X_i \\ \delta Y_i \\ \delta Z_i \end{vmatrix} + \begin{vmatrix} \delta X_j \\ \delta Y_j \\ \delta Z_j \end{vmatrix} - \Delta U_r \begin{vmatrix} \delta\omega_X \\ \delta\omega_Y \\ \delta\omega_Z \end{vmatrix} - \delta_k \begin{vmatrix} \Delta X_{ij} \\ \Delta Y_{ij} \\ \Delta Z_{ij} \end{vmatrix} + f_{\Delta X, \Delta Y, \Delta Z} \qquad (2.60)$$

where

$$f_{\Delta X, \Delta Y, \Delta Z} = \begin{vmatrix} \Delta X_{ij} \\ \Delta Y_{ij} \\ \Delta Z_{ij} \end{vmatrix} - \begin{vmatrix} \Delta X^{obs}_{ij} \\ \Delta Y^{obs}_{ij} \\ \Delta Z^{obs}_{ij} \end{vmatrix} . \qquad (2.61)$$

Using equation (2.58), we transform the C.T. coordinate shifts (unknowns) to the local geodetic systems at station i and j; details of the resulting equations are given in section 4.

2.3 Observation Equations for Inertial Survey Observations

We will consider smoothed ($\Delta\phi$, $\Delta\lambda$) as observations. The Δh component in ISS networks is smoothed separately, based on different control than that used for the ($\Delta\phi$, $\Delta\lambda$) smoothing and therefore can be treated separately; we will not consider the Δh further here but will assume that the corresponding height information will be treated separately and used in the readjustment as supplementary information rather than as observations.

We assume here that latitude and longitude differences are given in the form ($\Delta\phi_{1,i}$, $\Delta\lambda_{1,i}$,) i.e. with respect to an arbitrary station 1. The basic observation equations in the ellipsoidal system are

$$\phi_i - \phi_1 - \Delta\phi_{1,i} = 0 \tag{2.62}$$

$$\lambda_i - \lambda_1 - \Delta\lambda_{1,i} = 0 \tag{2.63}$$

and their linearized forms are given by

$$r_{\Delta\phi} = \delta\phi_i - \delta\phi_1 + f_{\Delta\phi} \tag{2.64}$$

$$r_{\Delta\lambda} = \delta\lambda_i - \delta\lambda_1 + f_{\Delta\lambda} \tag{2.65}$$

with

$$f_{\Delta\phi} = \phi_i - \phi_1 - \Delta\phi_{1,i}^{obs} \tag{2.66}$$

$$f_{\Delta\lambda} = \lambda_i - \lambda_1 - \Delta\lambda_{1,i}^{obs} \tag{2.67}$$

where ϕ_i, λ_i, ϕ_1, λ_1 are the up-to-date values of ellipsoidal coordinates of stations i and 1 respectively.

In order to transform equations (2.66) and (2.67) to the local geodetic systems, we make use of the relations

$$\delta\phi_i = m_i \cdot \delta x_i \tag{2.68}$$

$$\delta\lambda_i = n_i \cdot \delta y_i \tag{2.69}$$

where

$$m_i = (\rho_i + h_i)^{-1} \tag{2.70}$$

$$n_i = ((\nu_i + h_i)\cos\phi_i)^{-1} \tag{2.71}$$

(with $\rho_i = a(1-e^2)/(1-e^2\sin^2\phi_i)^{3/2}$).

Substituting these in equations (2.66) and (2.67) gives

$$r_{\Delta\phi} = m_i\,\delta x_i - m_1\,\delta x_1 + f_{\Delta\phi} \tag{2.72}$$

$$r_{\Delta\lambda} = n_i\,\delta y_i - n_1\,\delta y_1 + f_{\Delta\lambda} \quad . \tag{2.73}$$

Since in the smoothing of the ISS data the existing coordinates of control stations are used as fixed constraints, the scale and orientation (in azimuth) of these control stations are directly imbedded in the ISS observations. Since this scale and azimuth information will be included in the adjustment via the observations which, independent of the ISS, define that control, we must remove this information from the ISS observations in the adjustment by assigning scale and orientation unknowns to them. Also, since in the smoothing the difference in scale in the north-south and east-west directions is modelled, we need only solve for one scale factor. In order to introduce scale and azimuth orientation unknowns in equations (2.72) and (2.73), we make use of the similarity transformation in terms of local geodetic coordinates, which, when simplified for a small scale change k and a small azimuth rotation α, is given by

$$\begin{vmatrix} x^1 \\ y^1 \end{vmatrix} \doteq (1+k) \begin{vmatrix} 1 & \alpha \\ -\alpha & 1 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} \quad . \tag{2.74}$$

Carrying out the multiplication and dropping second order terms gives

$$\begin{vmatrix} x^1 \\ y^1 \end{vmatrix} \doteq \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} x \cdot k + y \cdot \alpha \\ y \cdot k - x \cdot \alpha \end{vmatrix} \tag{2.75}$$

and differentiation of this gives

$$\delta x^1 = \delta x + x \cdot \delta k + y \cdot \delta\alpha \tag{2.76}$$

$$\delta y^1 = \delta y + y \cdot \delta k - x \cdot \delta\alpha \quad . \tag{2.77}$$

Substituting δx^1 and δy^1 for their corresponding terms in equations (2.72) and (2.73) gives

$$r_{\Delta\phi} = m_i\,\delta x_i - m_1\,\delta x_1 + (m_i x_i - m_1 x_1)\,\delta k + (m_i y_i - m_1 y_1)\,\delta\alpha + f_{\Delta\phi} \tag{2.78}$$

$$r_{\Delta\lambda} = n_i\,\delta y_i - n_1\,\delta y_1 + (n_i y_i - n_1 y_1)\,\delta k + (-n_i x_i + n_1 x_1)\,\delta\alpha + f_{\Delta\lambda} \tag{2.79}$$

By choosing station 1 as the origin of the local geodetic system to which the (x_i, y_i) refer (for purposes of computing scale and rotation unknowns), x_1 and y_1 are identically zero. This will simplify equations (2.78) and (2.79) to

$$r_{\Delta\phi} = m_i \delta x_i - m_1 \delta x_1 + (m_i x_i)\delta k + (m_i y_i)\delta\alpha + f_{\Delta\phi} \quad (2.80)$$

and

$$r_{\Delta\lambda} = n_i \delta y_i - n_1 \delta y_1 + (n_i y_i)\delta k - (n_i x_i)\delta\alpha + f_{\Delta\lambda} \quad . \quad (2.81)$$

3. AUXILIARY PARAMETERS

In this section we give a summary of the "auxiliary parameters" to be included in the readjustment. By "auxiliary parameters" is meant all unknown parameters except station coordinate shifts δx, δy, δh. We have already discussed the scale and orientation unknowns for the ISS observations and the scale, orientation and translation unknowns for satellite Doppler and VLBI observations in section 2. In this section we will complete these discussions and will discuss the auxiliary parameters for the terrestrial observations.

Horizontal unoriented direction observations can be considered as angles measured from a line of unknown azimuth to lines to surrounding stations. This unknown azimuth, or "orientation unknown" must therefore be included in the direction observation equation as an auxiliary parameter. For each independently measured set of directions we have one orientation unknown Ω. In the readjustment these orientation unknowns will be solved for explicitly, i.e. they will not be eliminated a priori using, for example, the Schreiber method. Instead, the elimination will be done at the first level in the Helmert blocking method. The basic observation equation for directions becomes (compare with equation (2.1))

$$f_d = \arctan \frac{v_{ij}}{u_{ij}} - d_{ij} - \Omega = 0 \quad (3.1)$$

and its linearized form is detailed in section 4.

A group of astronomic azimuth observations with common unknown systematic error in orientation can be assigned an orientation unknown α. The corresponding observation equation is given by

$$f_A = \arctan \frac{v_{ij}}{u_{ij}} - A_{ij} - \alpha = 0 \quad (3.2)$$

and its linearized form is detailed in section 4.

Distance observations (Geodimeter, Tellurometer and Aerodist measurements) may contain systematic errors of two types which may have to be modelled. These are i) a scale error and ii) a constant error (or "zero correction").

Denoting the difference in scale (from the scale as given in the adjustment by other observations) by k and the constant error by c, the basic observation equation, when both of these parameters are included, is

$$f_S = \{(X_j-X_i)^2+(Y_j-Y_i)^2+(Z_j-Z_i)^2\}^{1/2} - S_{ij}(1+k) + c = 0 \qquad (3.3)$$

and its linearized form is detailed in section 4.

We will now discuss the various auxiliary parameters in the context of relating the adjusted station coordinates to the chosen reference coordinate system (i.e. the C.T. system). To define this relationship, we must supply information to the adjustment, via the observation equations, about the three basic quantities of position (i.e. the origin of the C.T. system), orientation and scale.

For defining "absolute" position, we have only one observation type, namely the satellite Doppler based on the precise ephemeris. For these observations, we therefore omit the δX_o, δY_o, δZ_o parameters from their observation equation (equation 4.). Any known systematic error in absolute position of these Doppler coordinates will be applied before they are input to the adjustment. Since the satellite Doppler coordinates which are based on the broadcast ephemeris are relatively inaccurate in absolute position, they will be treated as coordinate difference observations (which removes their inaccurate absolute position information content).

We have two basic sources of observational information for defining orientation, namely the astronomic observations of latitude, longitude and azimuth and the satellite Doppler coordinates based on the precise ephemeris. To a certain extent, the decision as to which should be used to define orientation is arbitrary since, from this point of view, they are of equal accuracy. (We do not wish to allow both of these observation types to supply orientation information to the adjustment since they are referred to different coordinate systems and any discrepancy in orientation between them would unduly cause an increase in the magnitudes of their residuals. This also applies to scale information which we discuss next). However, the observation equations resulting from allowing the astronomic observations to define orientation are simpler (see Vincenty, 1982b). This is the approach to be used in the readjustment and therefore orientation unknowns ω_X, ω_Y and ω_Z will be included in the satellite Doppler (and VLBI) observation equations.

Regarding scale, in Canada there are five possible independent sources: VLBI, satellite Doppler, Geodimeter, Tellurometer and Aerodist observations. If the VLBI data is prepared in time for the readjustment, it will provide the most accurate scale information. However, the precise ephemeris satellite Doppler coordinates provide a more homogeneous spatial distribution of scale control. Also, from a continental point of view, the precise ephemeris Doppler coordinates provide more accurate scale information than either the Geodimeter, Tellurometer or Aerodist observations.

Therefore, (separate) scale unknowns will be included in the observation equations for Geodimeter, Tellurometer and Aerodist observations. The VLBI and precise ephemeris Doppler observations will not have scale unknowns assigned to them and will therefore provide scale information to the adjustment (after scale consistency between these two observation types is ensured through prior intercomparisons). Also, a scale unknown will be assigned to each independent group of satellite Doppler coordinate difference observations based on the broadcast ephemeris.

Table 3.1 gives a summary of auxiliary parameters to be included in the readjustment.

Observation Type	Auxiliary Parameters			
	Translation	Orientation	Scale	Constant
directions	-	Ω	-	-
Geodimeter	-	-	k	c
Tellurometer	-	-	k	c
Aerodist	-	-	k	c
azimuths	-	*, α	-	-
precise Doppler	*	ω_X, ω_Y, ω_Z	*	-
broadcast Doppler	-	ω_X, ω_Y, ω_Z	k	-
VLBI	-	ω_X, ω_Y, ω_Z	*	-
ISS	-	α	k	-

* - provides corresponding information to adjustment

TABLE 3.1 SUMMARY OF AUXILIARY PARAMETERS

4. SUMMARY OF OBSERVATION EQUATION COEFFICIENTS FOR COMPUTER IMPLEMENTATION

This section is given as a quick reference for programming use as well as to complete the detail of the observation equations as referred to the local geodetic coordinate systems.

4.1 General

Preliminary station coordinates are given as geodetic latitude ϕ, geodetic longitude λ and ellipsoidal height h, all referred to the geocentric reference ellipsoid. For use in computing observation equation coefficients, the geocentric Cartesian coordinates X, Y, Z (in the conventional terrestrial (C.T.) system) are computed as follows for each station:

$$X_i = (\nu_i + h_i)\cos\phi_i \cos\lambda_i \tag{4.1}$$

$$Y_i = (\nu_i + h_i)\cos\phi_i \sin\lambda_i \tag{4.2}$$

$$Z_i = (\nu_i (1-e^2) + h_i)\sin\phi_i \tag{4.3}$$

where

$$\nu_i = a/(1-e^2\sin^2\phi_i)^{1/2} \tag{4.4}$$

and

$$e^2 = (a^2-b^2)/a^2 \tag{4.5}$$

Also needed in computing the coefficients is the equatorial distance

$$p_i = (X_i^2+Y_i^2)^{1/2} \ . \tag{4.6}$$

For each station we also need the astronomic latitude Φ_i and astronomic longitude Λ_i. If these were measured at a station then these measured values must be used. For stations where they were not measured, we compute them from

$$\Phi_i = \phi_i + \xi_i \tag{4.7}$$

$$\Lambda_i = \lambda_i + \eta_i/\cos\phi_i \tag{4.8}$$

where ξ_i, η_i are the meridian and prime vertical surface deflections of the vertical. Note that both the observed Φ, Λ (corrected for polar motion) and the deflections ξ, η are surface quantities. For use in the computation of observation equation coefficients the sines and cosines of Φ, Λ should be computed and stored in a file which is maintained until the adjustment is finished. In this way, these sines and cosines can be updated after each iteration by

$$\sin\Phi_{new} = \sin\Phi_{old} + \cos\Phi_{old} \cdot \delta\phi$$

$$\cos\Phi_{new} = \cos\Phi_{old} - \sin\Phi_{old} \cdot \delta\phi$$

$$\sin\Lambda_{new} = \sin\Lambda_{old} + \cos\Lambda_{old} \cdot \delta\lambda \qquad (4.9)$$

$$\cos\Lambda_{new} = \cos\Lambda_{old} - \sin\Lambda_{old} \cdot \delta\lambda$$

(where $\delta\phi$ and $\delta\lambda$ are corrections computed during the adjustment as below) so that they need only be computed once with library routines. This update is, of course, done only for those stations for which there is no observed Φ, Λ.

The adjustment will result in corrections δx_i, δy_i, δh_i to local geodetic coordinates of the stations; these can be converted to corrections to ϕ_i, λ_i by

$$\delta\phi_i = \delta x_i/(\rho_i + h_i) \qquad (4.10)$$

$$\delta\lambda_i = \delta y_i \cdot \sec\phi_i/(\nu_i + h_i) \qquad (4.11)$$

where $\rho_i = a(1-e^2)/(1-e^2\sin^2\phi_i)^{3/2}$.

The δh unknown will only be assigned to stations for which three dimensional Cartesian coordinates are observed (i.e. satellite Doppler) or between which three dimensional coordinate differences are observed (i.e., satellite Doppler and VLBI).

4.2 Observation Equations for Terrestrial Observations.

The "terrestrial" observations of distance, azimuth and unoriented direction are discussed in this section. For each of these observations, the following quantities are used for computing coefficients:

$$c_i = -\sin\phi_i(\cos\Lambda_i\Delta X_{ij} + \sin\Lambda_i\Delta Y_{ij}) + \cos\Phi_i\Delta Z_{ij} \qquad (4.12)$$

$$d_i = -\sin\Lambda_i\Delta X_{ij} + \cos\Lambda_i\Delta Y_{ij} \qquad (4.13)$$

$$S^2_{ij} = \Delta X^2_{ij} + \Delta Y^2_{ij} + \Delta Z^2_{ij} \qquad (4.14)$$

where the subscripts i and j refer to the stations from and to which the observation is made, and

$$\Delta X_{ij} = X_j - X_i$$

$$\Delta Y_{ij} = Y_j - Y_i \qquad (4.15)$$

$$\Delta Z_{ij} = Z_j - Z_i \ .$$

Also, the factor

$$m_i = \sin\phi_i(\sin\Lambda_i \frac{X_i}{p_i} - \cos\Lambda_i \frac{Y_i}{p_i}) \qquad (4.16)$$

is needed for computing the coefficients.

4.2.1 Terrestrial Distance Observations

Distance observations are input as spatial straight-line lengths from one station marker to another (i.e., "marker-to-marker" distances). The linearized observation equation in the general case for these distances is given by (r_s is the distance residual):

$$r_s = a_1\delta x_i + a_2\delta y_i + a_3\delta x_j + a_4\delta y_j - S^0\delta k - \delta c + f_s \qquad (4.17)$$

where δk is the correction to the scale difference and δc is the correction to the "zero-correction" (constant error in the distance measurement). The coefficients a_i are given by

$$a_1 = -(c_i + m_i d_i)/S^0$$

$$a_2 = -(d_i - m_i c_i)/S^0 \qquad (4.18)$$

$$a_3 = -(c_j + m_j d_j)/S^0$$

$$a_4 = -(d_j - m_j c_j)/S^0$$

where $S^0 = (S^2_{ij}/S_{obs} + S_{obs})/2$ (where $S^2_{ij} = \Delta X^2_{ij} + \Delta Y^2_{ij} + \Delta Z^2_{ij}$). S_{obs} is the observed distance.

The misclosure f_s is given by

$$f_s = S^0 - S_{obs} - (1+k^0)c^0 - k^0 S^0 \qquad (4.19)$$

where c^0, k^0 are the most recent values of the "zero-correction" and the scale difference respectively. The weight for the distance observation equation is given by $1/\sigma^2_s$ where σ^2_s is the variance of the distance observation in units of metres squared.

4.2.2 Unoriented Direction Observations

Directions as observed are input (i.e., the mean of whatever number of sets were taken or the results of a station adjustment); no reductions (e.g., to the reference ellipsoid) are made. The linearized observation equation is given by

$$r_{dir} = a_1\delta x_i + a_2\delta y_i + a_3\delta x_j + a_4\delta y_j - \rho''\cdot\delta\Omega + f_{dir} \tag{4.20}$$

where $\delta\Omega$ is the correction to the orientation unknown Ω for the corresponding independent set of directions and ρ'' = 206264.806. For the computation of coefficients a_i, we first compute,

$$D_i^2 = (c_i^2+d_i^2) \tag{4.21}$$

$$\Delta\Lambda = \Lambda_j-\Lambda_i \tag{4.22}$$

$$q_1 = -d_i(\cos\phi_i\cos\phi_j+\sin\phi_i\sin\phi_j\cos\Delta\Lambda) - c_i\sin\phi_j\sin\Delta\Lambda \tag{4.23}$$

$$q_2 = c_i\cos\Delta\Lambda-d_i\sin\phi_i\sin\Delta\Lambda \tag{4.24}$$

and then,

$$\begin{aligned} a_1 &= (d_i-m_ic_i)/D_i^2 \\ a_2 &= -(c_i+m_id_i)/D_i^2 \\ a_3 &= (q_1+m_jq_2)/D_i^2 \\ a_4 &= (q_2-m_jq_1)/D_i^2 \end{aligned} \tag{4.25}$$

The orientation unknowns will not be eliminated (e.g., via the Schreiber method) but will be solved for as will all other nuisance parameters. The units of $\delta\Omega$ in the solution vector will be radians, the initial value Ω^0 can be computed using

$$\Omega^0 = \arctan(d_i/c_i)-d_{R.O.} \tag{4.26}$$

where d_i and c_i are the values for the "reference origin" (R.O.) direction (that direction in the set with the smallest numerical value) and $d_{R.O.}$ is the observed value of that direction. The misclosure f_d is given by (which must be computed in units of arc-seconds)

$$f_d = \arctan(d_i/c_i)-d_{obs}-\Omega^0 \tag{4.27}$$

where d_{obs} is the observed direction and Ω^0 is the most up-to-date value of the orientation unknown. The weight for a direction is given by $1/\sigma^2_{dir}$ where σ^2_{dir} is the variance of the direction in units of arc-seconds squared.

4.2.3 Astronomic Azimuth Observations

Azimuth observations are assumed to be corrected for polar motion. They are not to be reduced to geodetic azimuths (e.g., the deflection or skew-normal corrections are not made). The linearized observation equation is given by

$$r_{Az} = a_1\delta x_i + a_2\delta y_i + a_3\delta x_j + a_4\delta y_j - \rho''\delta\alpha + f_{Az} \tag{4.28}$$

where $\delta\alpha$ is the correction for the azimuth orientation unknown α (this unknown may be specified for a group of azimuths with a suspected systematic error in orientation).

The coefficients a_i are the same as for unoriented directions (see previous section).

The misclosure f_{Az} is given by

$$f_{Az} = \arctan(d_i/c_i) - Az_{obs} - \alpha \tag{4.29}$$

where α is the most recent value for the orientation unknown (whose initial value may be taken as zero). Note that the units of $\delta\alpha$ will be in radians. The units of f_{Az} must be arc-seconds. The weight of the observed azimuth is given as $1/\sigma^2_{Az}$ where σ^2_{Az} is the variance of the observed azimuth in units of arc-seconds squared.

4.3. Observation Equations for Satellite Doppler and VLBI

In the case of satellite Doppler, we consider both coordinate and coordinate difference observations. The coordinate difference observation equations are identical for Doppler coordinate differences and VLBI. It is assumed that, for both Doppler and VLBI, coordinates or coordinate differences are referred to the Conventional Terrestrial (geocentric, three-dimensional, Cartesian) system. That is, it is assumed that known corrections (e.g., the two to four metre Z-shift for Doppler coordinates) are applied.

4.3.1 Coordinate Observations

The observation equations for a set of C.T. coordinate observations for a group of n stations, for which a full covariance matrix is supplied, is of the form

$$r = A\delta + f\ . \tag{4.30}$$

where $r = [r_{X_1}\ r_{Y_1}\ r_{Z_1}\ r_{X_2}\ r_{Y_2}\ r_{Z_2} \cdots r_{X_n}\ r_{Y_n}\ r_{Z_n}]^{\dagger}$.

Let

$$\delta_i = [\delta_{x_i} \ \delta_{y_i} \ \delta_{h_i}] \tag{4.31}$$

$$\delta_o = [\delta X_o \ \delta Y_o \ \delta Z_o] \tag{4.32}$$

$$\delta_\omega = [\delta\omega_X \ \delta\omega_Y \ \delta\omega_Z] \tag{4.33}$$

where X_o, Y_o, Z_o are geocentric translation components and ω_X, ω_Y, ω_Z are small rotations around the X, Y, Z axes respectively. Let δk denote the correction for the scale difference k.

Also let

$$F_i = \begin{vmatrix} -\sin\phi_i\cos\lambda_i & -\sin\lambda_i & \cos\phi_i\cos\lambda_i \\ -\sin\phi_i\sin\lambda_i & \cos\lambda_i & \cos\phi_i\sin\lambda_i \\ \cos\phi_i & 0 & \sin\phi_i \end{vmatrix} \tag{4.34}$$

$$G_i = \begin{vmatrix} 0 & -Z_i & Y_i \\ Z_i & 0 & -X_i \\ -Y_i & X_i & 0 \end{vmatrix} \tag{4.35}$$

$$-I = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \tag{4.36}$$

and

$$R_i = \begin{vmatrix} X_i \\ Y_i \\ Z_i \end{vmatrix} \tag{4.37}$$

where X_i, Y_i, Z_i are the up-to-date C.T. coordinates of the i^{th} station.

Then,

$$A = \begin{vmatrix} F_1 & 0 \cdots\cdots 0 & -I & -G_1 & -R_1 \\ 0 & F_2 \cdots\cdots 0 & -I & -G_2 & -R_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots\cdots F_n & -I & -G_n & -R_n \end{vmatrix} \tag{4.38}$$

$$\delta = [\delta_1 \quad \delta_2 \cdots \delta_n \quad \delta_o \quad \delta_\omega \quad \delta_k] \tag{4.39}$$

and, letting

$$\bar{R}_i = R_i - \begin{vmatrix} X_i^{obs} \\ Y_i^{obs} \\ Z_i^{obs} \end{vmatrix} \tag{4.40}$$

we have

$$f = \begin{vmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \vdots \\ \bar{R}_n \end{vmatrix} \tag{4.41}$$

where X_i^{obs}, Y_i^{obs}, Z_i^{obs} are the observed coordinates of i^{th} station.

4.3.2 Coordinate Difference Observations

The $3(n-1)$ coordinate difference observations for a group of n stations will be considered as given in the form (for which a full covariance matrix is supplied)

$$[\Delta X_{1,2} \ \Delta Y_{1,2} \ \Delta Z_{1,2} \ \Delta X_{1,3} \ \Delta Y_{1,3} \ \Delta Z_{1,3} \cdots \Delta X_{1,n} \ \Delta Y_{1,n} \ \Delta Z_{1,n}]^t$$

where the coordinate differences are defined as

$$\Delta X_{i,j} = X_j - X_i$$

$$\Delta Y_{i,j} = Y_j - Y_i \qquad (4.42)$$

$$\Delta Z_{i,j} = Z_j - Z_i \ .$$

The observation equations are of the form

$$r = A\delta + f \qquad (4.43)$$

where

$$r = [r_{\Delta X_{1,2}}\ r_{\Delta Y_{1,2}}\ r_{\Delta Z_{1,2}}\ r_{\Delta X_{1,3}}\ r_{\Delta Y_{1,3}}\ r_{\Delta Z_{1,3}} \cdots r_{\Delta X_{1,n}}\ r_{\Delta Y_{1,n}}\ r_{\Delta Z_{1,n}}] \ .$$

With δ_i, δ_ω, δk and F_i defined as in the previous section and letting

$$\Delta G_{i,j} = \begin{vmatrix} 0 & -\Delta Z_{ij} & \Delta Y_{ij} \\ \Delta Z_{ij} & 0 & -\Delta X_{ij} \\ -\Delta Y_{ij} & \Delta X_{ij} & 0 \end{vmatrix} \qquad (4.44)$$

$$\Delta R_{i,j} = \begin{vmatrix} \Delta X_{i,j} \\ \Delta Y_{i,j} \\ \Delta Z_{i,j} \end{vmatrix} \qquad (4.45)$$

where ΔX_{ij}, ΔY_{ij}, ΔZ_{ij} are computed from the up-to-date values of X, Y, Z,

we have,

$$A = \begin{vmatrix} -F_1 & F_2 & 0 & 0 \ldots 0 & -\Delta G_{1,2} & -\Delta R_{1,2} \\ -F_1 & 0 & F_3 & 0 \ldots 0 & -\Delta G_{1,3} & -\Delta R_{1,3} \\ \vdots & \vdots & & \ddots \quad \vdots & \vdots & \vdots \\ -F_1 & 0 & \ldots\ldots & F_n & -\Delta G_{1,n} & -\Delta R_{1,n} \end{vmatrix} \tag{4.46}$$

and

$$\delta = [\delta_1 \delta_2 \ldots \delta_n \delta_\omega \delta_k] \,. \tag{4.47}$$

Letting

$$\Delta\bar{R}_{ij} = \Delta R_{i,j} - \begin{vmatrix} \Delta X^{obs}_{i,j} \\ \Delta Y^{obs}_{i,j} \\ \Delta Z^{obs}_{i,j} \end{vmatrix} \tag{4.48}$$

we have,

$$f = \begin{vmatrix} \Delta\bar{R}_{1,2} \\ \Delta\bar{R}_{1,3} \\ \vdots \\ \Delta\bar{R}_{1,n} \end{vmatrix} \tag{4.49}$$

where $\Delta X^{obs}_{i,j}$, $\Delta Y^{obs}_{i,j}$, $\Delta Z^{obs}_{i,j}$ are the observed coordinate differences.

5. REFERENCES

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