Guidelines for the Integration of Geodetic Networks in Canada

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The readjustment of the North American horizontal geodetic networks is scheduled for late 1985. In Canada, this will provide new coordinates and covariance matrix for a relatively sparse framework consisting mainly of primary triangulation and Doppler satellite networks augmented by some secondary networks. The bulk of the secondary and lower order networks will be integrated with this framework shortly after the readjustment. This paper considers the approaches which can be taken to integration and addresses some of the problems that must be overcome. A step-by-step procedure is offered which is intended to promote uniformity and consistency among the various agencies involved in secondary integration. These guidelines are aimed at achieving an effective compromise between the requirement for maintaining a fairly static set of published coordinates and the requirement to minimize the effects of distortions due to significant misfits between existing control and surveys to be integrated.

La recompensation continentale des réseaux géodésiques planimétriques de l'Amérique du Nord est attendue vers la fin de l'année 1985. Au Canada, cette compensation fournira aux utilisateurs un ensemble de nouvelles coordonnées avec leurs matrices de covariance pour la majeure partie du canevas national comprenant les réseaux de triangulation primaires, les points de satellite Doppler et certains réseaux secondaires. L'intégration de la plupart des réseaux planimétriques secondaires et d'ordre inférieur se fera après l'achèvement de cette nouvelle compensation des réseaux fondamentaux. L'article décrit les démarches à suivre lors de l'intégration et met en évidence quelques-uns des problèmes à surmonter. Aux diverses agences responsables de l'intégration, une procédure systématique est offerte afin de promouvoir l'uniformité et une approche logique de l'intégration des réseaux secondaires. Ces directives sont dressées dans le seul but d'atteindre un compromis réaliste entre le besoin de maintenir un service sûr des valeurs géodésiques publiées et le besoin de minimiser les influences des distorsions dues aux différences appréciables entre les contrôles existants et les levés à être intégrés.

INTRODUCTION

The readjustment of the North American horizontal geodetic networks is scheduled for late 1985. At that time, a new set of adjusted coordinates and corresponding covariance matrix will be available for a framework of control stations.

The Canadian framework will consist mainly of primary triangulation and Doppler satellite networks, augmented by some secondary networks. The bulk of the secondary and lower order networks will be included in a secondary integration, to be completed shortly after the framework readjustment.

This paper will attempt to define guidelines for integration which will promote uniformity and consistency among the various agencies involved in secondary integration.

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Integration is considered in the broader sense, as these guidelines must be applicable to all future additions, deletions or changes to the geodetic networks.

The problem with integrating surveys to an already existing, adjusted network is that changes to the coordinates and covariance matrix of the existing network are implied. That is, if the additional surveys had been included in the original network adjustment, the coordinates and covariance matrix of the network stations would be, in general, different from their existing values. The magnitudes of the differences would depend on how well the two networks "fit together" (i.e., on the magnitudes of the systematic errors inherent in the networks).

The objectives of integration, then, are to obtain a merger that is as free as possible of distortions due to statistically significant misfits, and to have all the information available after the merger needed for future work (i.e., future integration, error analysis, station classification, etc.). While the objectives are constrained in that published coordinates must be as static as possible, distortions must not be allowed to adversely affect either the control or integrated surveys. Obviously, a compromise must be reached between truly static coordinates and distortionless integration.

The considerations for maintaining a minimally distortion-free set of coordinates will also apply to the associated covariance matrix. The covariance matrix of the integrated survey, as well as any significant changes to the covariance matrix of the control, must be maintained, since this information increases the usefulness of the merged networks and allows meaningful accuracy statements to be made. Again, a practical balance must be sought between the ideal and the requirement for static coordinates. See [*Chamberlain et al.* 1985] for a discussion of postreadjustment maintenance schemes.

Prior to discussing the details of integration, a clarification of some of the terminology is in order. The existing framework is referred to as "control" and the survey to be merged or integrated as "densification". "Coordinate residuals" are the result of an integration and are the differences between the previous control coordinates and their new estimates. "Misfits" are a measure of coordinate residuals, and imply that these residuals are in excess of statistical limits. Therefore, while integration will always produce coordinate residuals, we would expect that misfits would rarely occur.

Alternative Integration Methods

There are a number of integration procedures that can be employed in search of the optimum compromise between maintaining static coordinates and minimizing the propagation of distortions throughout the merged networks. Each has relative merits and failings, and are described in the following.

Dynamic Approach

Theoretically, the best approach to eliminating distortions due to misfits is to maintain a completely dynamic network. That is, every time a new survey is made, the entire national network, including the densification, is readjusted. Of course, this is entirely impractical and, fortunately, not necessary in order to keep distortions to a statistically acceptable level. This does not preclude the possibility of maintaining both a fairly static file of published coordinates and also a file of near-dynamic coordinates. The latter could be updated more frequently, providing a set of "best available" coordinates for scientific purposes. The dynamic file could be maintained without readjustment of the entire network each time a densification survey is made. Two of the alternatives to be discussed can be used to accomplish integration in a "rigorous" manner with only local networks being readjusted.

Weighted Station Approach

The weighted station method makes use of the covariance matrix of a "band" of control surrounding the densification. This band includes stations not directly connected to the densification, as well as the stations common to both networks. This covariance matrix is input, together with the control station coordinates and the densification observations, into a rigorous least squares adjustment. New coordinates and covariance information is then generated for all stations included. A detailed description of this method is presented in Appendix A.

The method may be summarized by:
Existing network:

$$\{\hat{x}; C_{\hat{x}}\} \equiv \{\ell_x; C_{\ell_x}\}$$

New observations:

$$\{\ell_1, \ell_2, \ldots, \ell_n; C_{\ell_1}, C_{\ell_2}, \ldots, C_{\ell_n}\} \equiv \{\ell; C_\ell\}$$

Weighted station method:

$$\{F(\ell, \hat{x}, \ell_x) = 0; C_\ell, C_{\ell_x}\}$$

The main advantage of the weighted station approach is that complete statistical information is generated for analysis purposes. It is expected that most control coordinate changes will be statistically insignificant. Thus, the static nature of the published coordinates can be maintained. However, where misfits occur, the weighted station approach will allow a complete analysis of the misfit, which may lead to its' reconciliation. Of course, the end result may mean a change in published coordinates. With a suitably selected band of existing control, the weighted station method will allow determination of the size of the published coordinates subset which must be altered. Criteria for selecting a control band size will be examined later.

Sequential Approach

The sequential adjustment approach is theoretically identical to the weighted station approach, from the adjustment point of view. It is rigorous, and will permit a complete statistical evaluation. However, the weighted station "batch" mode is, for a typical network integration, computationally more efficient than the "piecemeal" mode of the sequential adjustment. With the latter we again have:
Existing network:

$$\{\ell_x; C_{\ell_x}\}$$

New observations:

$$\{\ell; C_\ell\}$$

Sequential approach:

$$\hat{x}_i = g[F(\ell_i, x, \ell_{x_{i-1}}) = 0; C_{\ell_i}, C_{\ell_{x_{i-1}}}]$$

This method may be more useful in identifying causes of misfit since its main attribute is the manner in which portions of the densification or control networks may be sequentially added, and thus analyzed separately, during the integration.

Minimum Constraint Approach

This approach is by no means suggested as a method to carry out integration, but is very useful for the analysis of the densification surveys. The minimum constraint least squares adjustment is performed with a minimum of influence from the control. This influence is in the form of the elements necessary to overcome datum defects and avoid rank deficiency in the densification normal equations. Coordinate system origin, scale, and orientation are the elements to be considered. They may be provided from the control by fixing a control station, a distance, and an azimuth, respectively. These elements would be considered as errorless. Alternatively, the coordinates of two control stations can be held fixed. Of course, if the densification observations include one or more of these elements, then they will not be required from the control. For example, with traverses that contain direction, distance and azimuth observations, only one fixed control station is required.

The minimum constraint adjustment allows a complete statistical analysis of the densification itself, independent of the control. Therefore it provides a means to evaluate the densification survey and to identify weaknesses before merger with the control.

Fixed-Control Adjustment Approach

This approach is the other extreme when compared to the minimum constraint approach in that all the control coordinates are held fixed. That is, the densification survey is not allowed to affect the control. Neither the control coordinates nor their covariance matrix will change. Also referred to as the "over-constrained" method, this approach is not desirable (and *not recommended*) in that the densification is forced to fit the control, regardless of any distortion that exists in either network and regardless of their comparative accuracies. Misfits are not easily detected and a rigorous analysis of misfits is not possible since the statistical information is not generated.

The practice of holding control coordinates error-free in network integrations has been the major contributor to the existing distortions in the NAD27 network. Indeed, the impetus for carrying out NAD83 is the existence of these unacceptable distortions.

Transformation Approach

One other approximate method to be considered (but *not recommended*) is the transformation of the densification coordinates, based on the coordinate differences at common points between the control and the densification. This method also forces the densification to fit the control, regardless of distortions or the comparative accuracies of the two networks. Complete statistical analysis is not possible since the information required is not available. This approach would be useful when the control is significantly more accurate than the densification or the covariance information for the densification is missing or suspect. Examples of transformation methods are the usual similarity transforms, the complex polynomial method as used by program ESTPM at the Geodetic Survey of Canada, and a collocation approach developed there.

Blaha Approach

Another approach which has been considered but not recommended is that proposed by *Blaha* [1974]. Essentially, this is similar to the "fixed adjustment" approach in that the

control coordinates are not allowed to be altered by the densification. The major difference is that the control coordinates are not considered errorless and their covariance matrix is applied. The approach is characterized by:
Existing network:

$$\{\hat{x}, C_{\hat{x}}\} \equiv \{\ell_x; C_{\ell_x}\}$$

New observations:

$$\{x_N, \ell, C_\ell\}$$

where

$$F_1(x, x_N, \ell) = 0$$

$$F_2(x, \ell_x) = 0$$

and

$$x - \hat{x} = 0$$

While the approach is attractive from the viewpoint of static published coordinates, the absence of coordinate residuals, and therefore the detection and analysis of misfits, precludes its use.

Integration Considerations

Choosing an Approach

The recommended approach for integration is the weighted station adjustment. While the sequential adjustment is also acceptable, the weighted station approach is more advantageous, in general, from a computational efficiency point of view.

The majority of integration cases will involve the availability of the covariance information for both the control and the densification. It is for these cases that the general integration procedures outlined here are applicable. (However, we must also consider cases where this information is unavailable, or suspect, for either the control or the densification.) For the control, the information will be in the form of the covariance matrix, generated as a by-product of the continental readjustment. For the densification, this information can be either in the form of the original observations and their estimated variances, or adjusted coordinates and covariance matrix as obtained from a minimum constraint adjustment.

The absence of covariance information for the control is highly unlikely. However, if it should occur in some obscure case, we should not let the control, whose quality is unknown, to adversely affect the densification, whose quality can be estimated. Thus, the minimum constraint adjustment is preferred in this case. Obviously, a complete statistical analysis will not be possible, and this case has no place in proper integration procedures.

The absence of covariance information for the densification generally implies that it is of very low accuracy. One example might be the integration of old chain traverses, where the original coordinates are available, but not the observations. The transformation approach is best suited for this case.

Control Band Size Considerations

Selecting the size of the band of control stations is problematic in that no exact method, or rules of thumb, can be identified. Obviously, each integration case will require different considerations, such as comparative accuracies, types of observations, network strengths and reliabilities, etc. These considerations will relate to how far the effect of the densification is propagated into the surrounding control. Particular attention must be given to suspected weaknesses in the control, especially when the densification is strong in this area. For example, it has been shown that densification which is strong in scale will significantly affect a distant area of control which is weak in scale. *Carrière et al.* [1984] showed that when an error (blunder) was introduced to the densification distance observations, the effect on the control varied, not only with distance from the densification but also with the relative strengths of the networks involved. In one example, the discordance in the densification adversely affected an area of control remote from the densification and known to be weak in scale.

Therefore, it seems that selection of this band size must involve a "trial and error" process, with consideration given to the identified weaknesses in the local control network. Of course, one must first ensure that the densification is problem-free, by performing a minimum constraint adjustment and complete statistical analysis of the densification survey.

There are areas of the national framework which are suspected to be weak in scale and/or orientation. However, for the majority of the net, little is known about the network reliability. It is proposed that the integration process include examining the local control for its reliability using strain techniques. Strain analysis of geodetic networks is a method, developed by Dr. P. Vaníček and others, for the computation of strain elements from coordinate differences at a number of points. These elements give an indication of the local stretching (or compression) and rotation which is induced at each point due to surrounding coordinate differences. A much clearer picture of network distortions is the result. For details of the strain analysis approach, see *Thapa* [1980], *Vaníček et al.* [1981], and *Dare and Vaníček* [1982].

A research contract is ongoing between the Geodetic Survey of Canada and Marshall Macklin Monaghan Limited, Don Mills, Ontario to:

(1) develop methods of strain analysis for two- and three-dimensional geodetic networks;
(2) develop methods for interactively applying strain and error analysis techniques to two- and three-dimensional networks using CRT graphic displays;
(3) test different three-dimensional adjustment models in terms of network strength using strain analysis methods.

It is expected that delivery of an interactive graphics tool for strain and error analysis will be made in September, 1986.

Figure 1 represents a fictitious control net with traverse densification to be integrated. An initial band size is selected such that control points directly connected to the three control points to which the traverses are tied are included. This is probably a large enough band to start with. After performing the weighted station adjustment, coordinate residuals are examined for misfits. If they exist, the process is iterated using a larger band size — the next "layer" of control is added. Of course, before iterating the process, an extensive analysis is carried out to determine the cause of misfits.

If the strain analysis indicated problems in an adjoining section of control, this section would *not* be included in the band, since experience has shown that abnormally large residuals would occur in this section. If misfits occurred at the boundary of this section,

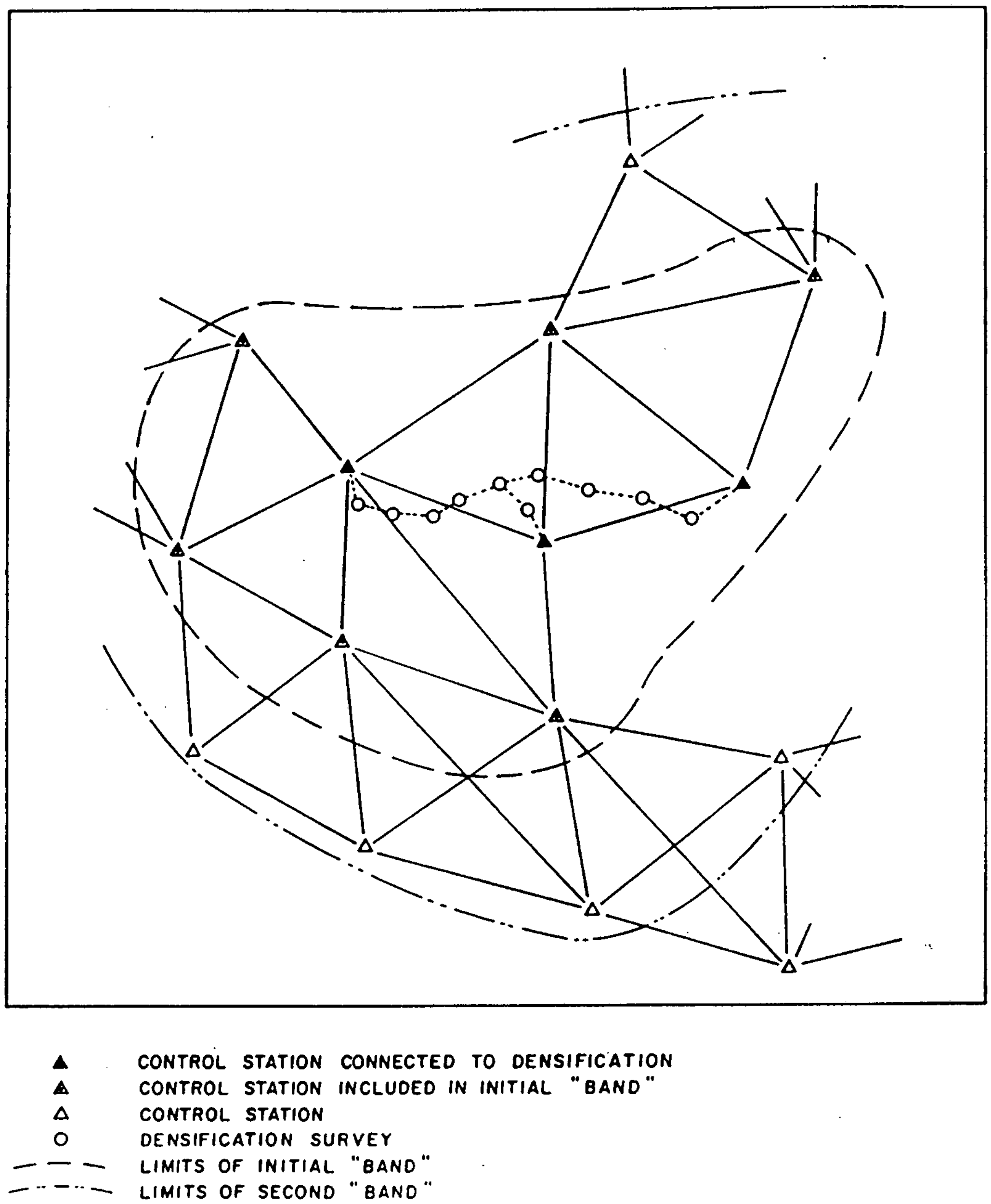

Figure 1. Bands of control.

then the only recourse would be to make additional observations to overcome the suspected control deficiencies.

Datum Problems

There have been suggestions that a proper integration process must take care of possible datum differences between the densification and the control. Generally, the control station coordinates define the coordinate system into which the densification is fitted. However, datum problems may still exist. For example, the scale defined by a precise GPS survey may not agree with the scale defined by tellurometer observations in the control.

Datum problems should be identified either in the reliability analysis of the control and densification, or in the analysis of the weighted station adjustment. In any case, if such problems are suspected, they can be modeled. See *Steeves* [1982] for a discussion of modeling auxiliary parameters.

Datum considerations are not expected to be a problem since we routinely account for them. For example, the "datum" difference between the geocentric coordinates from the Broadcast Doppler solution and the conventional Terrestrial Coordinate System (CT) is modeled using a seven-parameter similarity transformation. That is, the observed Doppler coordinates are transformed to the CT system by:

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix}_{CT} = \begin{vmatrix} X_0 \\ Y_0 \\ Z_0 \end{vmatrix} + kR \begin{vmatrix} X \\ Y \\ Z \end{vmatrix}_{OBS}$$

where X_0, Y_0 and Z_0 are translations, k is a scale factor, and R is a rotation matrix of rotations about the X, Y and Z axes. (See *Steeves* [1982]). This is just one example of how known datum problems can be overcome through the application of appropriate mathematical models.

Statistical Analysis of Adjustments

Analysis of Misfits

The proper integration of the densification survey will result in a new set of coordinates for the control stations. The coordinate residuals, differences between the new and original coordinates, must be analyzed for significance. There are two basic approaches, the statistical method and the strain analysis method.

The statistical method is based on defining statistical limits for the residual. If the residual is within this limit, we say it is not statistically significant. That is, we can correctly retain the published coordinate set. Of course, we must consider these residuals again for future integrations in the local area. This is a problem addressed in ongoing network maintenance (see *Chamberlain et al.* [1985]). If a misfit occurs (i.e., the residual is outside the prescribed limits), then we must attempt to determine its cause and take appropriate action to deal with it.

The two main statistical analyses which will be employed are residual analysis and reliability analysis. A concise summary of residual analysis models is presented in *Vaníček and Krakwisky* [1982]. In particular, tables 13.4 and 13.5 of this reference detail the χ^2 goodness-of-fit test of residuals, tests for residual outliers, etc. The determination of the internal and external reliability of a network is treated in *Kavouras* [1982], with emphasis on the detection of outliers.

The strain analysis technique can also be applied to the integration to analyze misfits. The technique is especially suited to the sequential adjustment approach. That is, by adding the densification piece by piece to the integration, the strain analysis method can give a clearer picture of the actual causes of misfit. The graphics analysis tool discussed in the section on band size may also provide a simple, but effective means of analyzing misfits.

Statistical Analysis of the Minimum Constraint Adjustment

The statistical analysis listed above will be used to evaluate the results of the minimum constraint adjustment. It is important to note that the objective, at this stage, is to determine the internal consistency (reliability) of the densification alone. That is, without considering the control network. While it is possible to examine the coordinate residuals between the existing values of control stations and their estimates from the minimum constraint adjustment, there are dangers in this approach. It is recommended that the weighted station adjustment be used to examine residual coordinate differences, as well as to provide information on possible datum problems between the control and the densification.

Statistical Analysis of the Weighted Station Adjustment

The equations for the weighted station adjustment are given in Appendix A, and the coordinate changes in the control are given by:

$$\Delta\hat{X} = \hat{X} - \ell_x \tag{1}$$

The hypothesis that $\Delta\hat{X}$ equals zero must be tested statistically. For this purpose, we will derive the covariance matrix $C_{\Delta\hat{X}}$. This is most efficiently done by using the well known expression for the covariance matrix of estimated residuals (see e.g., *Vaníček and Krakiwsky* [1982]).

That is:

$$\begin{aligned} C_{\Delta\hat{X}} &= \begin{vmatrix} C_{\hat{r}_\ell} & C_{\hat{r}_\ell \hat{r}_x} \\ (C_{\hat{r}_\ell \hat{r}_x})^T & C_{\hat{r}_x} \end{vmatrix} \\ &= \begin{vmatrix} C_\ell & 0 \\ 0 & C_{\ell_x} \end{vmatrix} - \begin{vmatrix} A_C & A_D \\ I & 0 \end{vmatrix} C_{\hat{X}} \begin{vmatrix} A_C^T & I \\ A_D^T & 0 \end{vmatrix} \\ &= \begin{vmatrix} C_\ell - AC_{\hat{X}}A^T & -A_C C_{\hat{X}_C} - A_D C^T_{\hat{X}_C \hat{X}_D} \\ -(C_{\hat{X}_C} A_C^T + C_{\hat{X}_C \hat{X}_D} A_\ell^T & C_{\ell_x} - C_{\hat{X}_C} \end{vmatrix} \end{aligned} \tag{2}$$

where $A = [A_C \; A_D]$, subscripts C and D refer to the control and densification, respectively, and the other terms are outlined in Appendix A. Therefore,

$$C_{\hat{r}_X} = C_{\Delta\hat{X}_C} = C_{\ell X} - C_{\hat{X}_C} \tag{3}$$

(see e.g., *Mikhail* [1976], p. 350).

Another approach to deriving $C_{\Delta\hat{X}_C}$ would be to apply the covariance law to equation (1) directly. In this case, we would first have to derive the covariance matrix $C_{\hat{X}_C \ell_X}$ of the vector $[\hat{X}_C^T, \ell_X^T]$ which is, in general, a full matrix. That is, $\hat{X}_C$ and ℓ_X are correlated, as is obvious.

Using $C_{\Delta\hat{X}_C}$ as given by equation (3) we can compute confidence regions for the elements of the vector $\Delta\hat{X}_C$. By comparing the elements of $\Delta\hat{X}_C$ with their confidence regions we can test the statistical hypothesis that they are equal to zero.

Integration Procedures

Figure 2 is a decision and process chart which outlines the major steps involved in the recommended integration process. The process is begun assuming that the densification observations have been fully preprocessed and edited, as necessary, and that sufficient control information is available to execute a [MINIMUM CONSTRAINT ADJUSTMENT]. Sufficient control information includes those elements needed to overcome datum defects in the adjustment. The control stations connected to the densification stations will be treated as unknowns in this adjustment (except for those control stations held fixed, if any).

The next step is to perform a [STATISTICAL ANALYSIS] of the adjustment results. The adjustment statistics, variance factor, residuals, point error ellipses, etc. are examined. The object is to examine the quality of the densification itself, without the influence of the control. That is, we want to ensure the densification is internally consistent and reliable. Based on the results of this analysis, the question [RESULTS OK?] can be addressed.

A NO response implies that problems were detected in the analysis stage. The source of the problem may be as simple as a blunder in the input data, improper relative weighting, etc. or may be a more serious problem involving systematic errors between the control and densification surveys. For example, there may be relative scale problems of the type noted in the discussion of datum problems. The next step is to [INVESTIGATE, IDENTIFY, EDIT]. That is, we attempt to determine the cause of the problem and to formulate and implement a solution. Analysis tools such as reliability analysis [*Kavouras* 1982], and strain analysis can be employed.

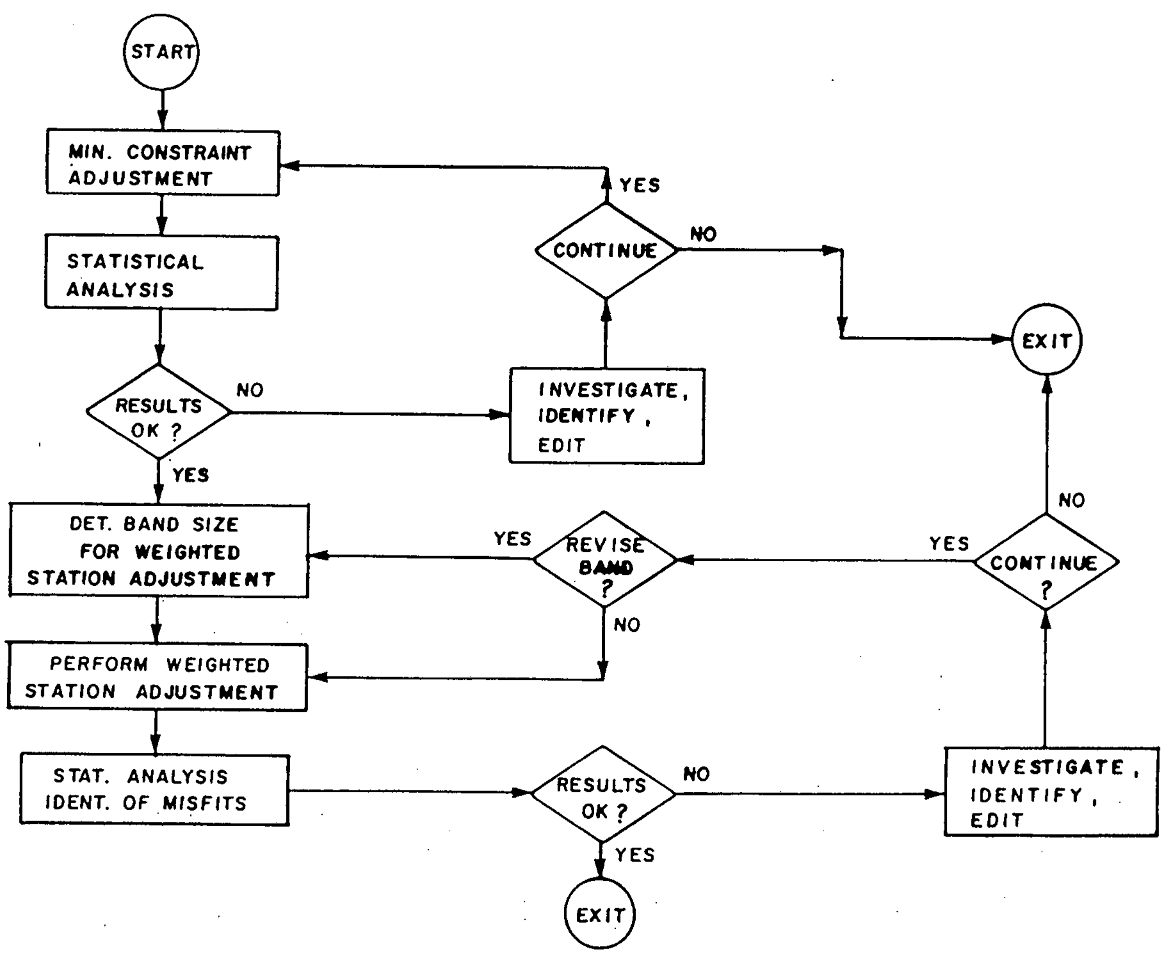

Figure 2. Integration process.

The results of this investigation provide the response to the next question, [CONTINUE?]. A NO response implies that the solutions to the problems require some external input; additional observations, for example. Therefore, we exit the process until such time as the required information is available to restart the integration.

A YES response triggers a second iteration of the minimum constraint adjustment and analysis loop. It may be necessary to reiterate this loop a number of times before an affirmative response to [RESULTS OK?] is generated. At this point, we should be satisifed that there are no significant systematic errors in the densification.

The next step is to [DET.BAND SIZE FOR WEIGHTED STATION ADJUSTMENT]. That is, the control stations to be included in the adjustment must be identified. Naturally, those control stations common to the densification will be included. In addition, control stations which are directly connected to these common control stations (via control network observations) will also be included. In this way, an initial band is selected. Refer to Figure 1 for an example. Of course, all control stations within this band would be included, along with their covariance matrices, in the weighted station adjustment. It is also at this point that we would examine the control network strength to identify areas of weakness. The interactive graphics tool using strain analysis would be one of the tools employed.

Having defined the band size, we next include the selected control stations, their covariance matrix, and the densification observations and [PERFORM WEIGHTED STATION ADJUSTMENT]. Next, we begin a [STAT. ANALYSIS; IDENT. OF MISFITS] process to determine if the coordinate residuals are statistically significant. We would also, of course, consider all the adjustment statistics in our analysis. Based on the analyses outputs, we can now decide how to respond to the question [RESULTS OK?].

A NO response initiates another [INVESTIGATE, IDENTIFY, EDIT] process. This implies that misfits have occurred, or some other problem has been noted in the analyses. We now attempt to determine the cause of the problem and to take whatever corrective action is possible. The results of this investigation lead to the next question, [CONTINUE?]. A NO response may be prompted by a decision to await additional observations, or perhaps simply because the required resources to continue are not immediately available. Whatever the reason, the NO decision results in exiting the process.

Having responded with YES, we must now decide whether to [REVISE BAND?]. A NO answer is given if the problems identified and corrected were not due to misfits. In this case, we proceed again to the weighted station adjustment. The existence of misfits would prompt a YES answer, with an extension of the band limits to the next "layer" of control. Again, special attention would be given to identified weak areas of the control. With the new band size, the loop continues again.

A number of iterations of this loop may be required before we arrive at an affirmative response to the [RESULTS OK?] question, and exit the integration process.

The [START] and two [EXIT] boxes indicate that the process is part of a more elaborate network maintenance scheme. In fact, Figure 2 is a detailed part of a much larger flowchart presented in *Chamberlain et al.* [1985].

CONCLUSION

An invariant set of published coordinates is not possible without allowing distortions to accumulate in the networks and, therefore, to quickly render NAD83 as problematic as we consider NAD27 to be today. Conversely, a truly dynamic set of coordinates is equally unacceptable. Therefore, we have attempted to outline procedures which strive to reach an optimum compromise between these two extremes. These procedures are but one process in the overall strategy of post readjustment network maintenance.

The weighted station method has been chosen as the primary adjustment tool for integration. The major reasons for its acceptance are the capability to provide complete statistical information, the ability to rigorously adjust a subset of the national network, and the ability to fully exploit the inherent strengths of both the control and densification surveys, as opposed to the traditional approach of always considering the control superior to the densification.

The guidelines are but a first step towards developing a viable integration process. Many questions and details remain to be resolved. However, if we wish to avoid the problems of NAD27 and maximize the life expectancy of NAD83, then we believe it is a step in the right direction.

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APPENDIX A

The Weighted Station Adjustment

The densification survey, in the form of a vector of observations, ℓ, and a covariance matrix, C_ℓ, is to be merged with an existing control network with known covariance matrix. A band of control stations surrounding the densification is selected. Note that this band will include stations not connected to the densification stations through the observations ℓ.

Let the existing coordinates of the control stations be denoted as the "observation" vector, ℓ_x, with covariance matrix C_{ℓ_x}. Denoting the "unknown" coordinates of these stations (the coordinates after integration) by x, we have:

$$x - \ell_x = 0 \tag{1}$$

whose linearized form (Taylor's series expansion) is given by:

$$A_{\ell_x}\delta_C + B_{\ell_x} r_x + w_x = 0 \tag{2}$$

Where A_{ℓ_x} and B_{ℓ_x} are design matrices whose elements are partial derivatives of equation (1) with respect to x_C and ℓ_x respectively, δ_C is the vector of unknown corrections to the initial values of x_C, r_x is the residual vector for ℓ_x and w_x is the misclosure vector [see e.g., Vaníček and Krakiwsky (1982)].

Similarly, we can write the linearized form of the observation equations for ℓ as:

$$A_C\delta_C + A_D\delta_D - r_\ell + w_\ell = 0 \tag{3}$$

where C and D subscripts refer to the control and densification, respectively.

Combining equations (2) and (3) and applying the least squares principle, we get:

$$\hat{\delta} = (A^T P A)^{-1} A^T P W \tag{4}$$

where,

$$\hat{\delta} = |\hat{\delta}_C, \hat{\delta}_D|^T \tag{5}$$

$$A = \begin{vmatrix} A_C & A_D \\ I & 0 \end{vmatrix} \tag{6}$$

$$P = \sigma_0^2 \begin{vmatrix} C_\ell^{-1} & 0 \\ 0 & C_{\ell_x}^{-1} \end{vmatrix} \tag{7}$$

and

$$w = |w_\ell, w_x|^T \tag{8}$$

The vector, $\hat{r}$, of estimated residuals is given by:

$$\hat{r} = |\hat{r}_\ell, \hat{r}_x|^T = A\hat{\delta} + w \tag{9}$$

The estimated parameters are given by:

$$\hat{x} = \begin{vmatrix} x_C^0 \\ x_D^0 \end{vmatrix} + \begin{vmatrix} \hat{\delta}_C \\ \hat{\delta}_D \end{vmatrix} \tag{10}$$

where $\hat{x}_C$ and x_D^0 are the most recent values for the parameters (in the iterative use of equations (4) and (10)). The associated covariance matrix is then:

$$C_{\hat{x}} = (A^T P A)^{-1} \tag{11}$$

where we assume the variance factor is known and equal to unity. This matrix can be partitioned as:

$$C_{\hat{x}} = \begin{vmatrix} C_{\hat{x}_C} & C_{\hat{x}_C\hat{x}_D} \\ (C_{\hat{x}_C\hat{x}_D})^T & C_{\hat{x}_D} \end{vmatrix} \tag{12}$$

We note that the coordinate changes of the control are given by:

$$\Delta \hat{x}_C = \hat{x}_C - \ell_x \tag{13}$$

which is also the estimated residual vector $\hat{r}_x$ of the observations ℓ_x. This can easily be shown by considering the expression for estimated residuals for the case of the linear observations equations. That is,

$$\begin{aligned} \hat{r}_x &= A_{\ell_x} \hat{x}_C - \ell_x \\ &= \hat{x}_C - \ell_x \end{aligned} \tag{14}$$

since $A_{\ell_x} = \mathrm{I}$.